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Path-Integral Representation for Tunneling Amplitudes in the Schwinger Model

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Feynman path-representations for tunneling amplitudes for both integer and half-integer winding configurations are derived in the Schwinger model. A comparison with a perturbation treatment is made.

1. INTRODUCTION

The discovery of the instanton solution [1] had profound implications in the understanding of the vacuum structure of gauge theories [2, 3, 4]. It was found that when massless fermions are coupled to the gauge fields, the resulting vacuum structure closely resembles the one known to exist in the Schwinger model [5]. At first sight the correspondence is not perfect because in the Schwinger model we do not have any Higgs fields, which would be needed to support the Nielsen-Oleson vortex [6] (the instanton in two dimensional space time). However, one should remember that the fermions in the Schwinger model induce an effective Higgs mechanism in the sense that the current is proportional to the vector potential. On this basis heuristic arguments were given [7] to the effect that regular vortices do play a role in the Schwinger model.

As has been shown by Nielsen and Schroer [8] the effective action in the Schwinger model obtained after integration of the fermion variables is rendered stationary by pairs of induced instantons-anti-instantons. These were shown to be responsible for the violation of the cluster property of correlation functions, thus implying a non trivial vacuum structure.

In this paper, in section 2, we carry the work of Ref. [8] a step further by deriving explicit Feynman-path representations for amplitudes between different vacua (Tunneling amplitudes) as integrals over configurations with non-trivial winding. In section 3 we discuss the results of the previous section from a perturbative point of view, clarifying their relation to the general mechanism proposed by 't Hooft [2]

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for massless fermions in the presence of winding fields. In section 4 we address ourselves to configurations with half-integer topological charge which play an important role in the confining property of the Schwinger model [9, 10].

2. Feynman Path Representation for The Tunneling Green's Functions

Although it is well known [8] that the vacuum structure in the Schwinger model, analyzed via clustering arguments, can be associated with induced instantons in this model, the Feynman path representation for the euclidean tunneling Green's functions, i.e.,

$$G^{[n]}(x, y) = \langle n \mid \psi(x_1) \cdots \psi(x_n)\bar{\psi}(y_1) \cdots \bar{\psi}(y_n) \mid 0 \rangle \tag{2.1}$$

have not been written down. In this section we arrive at such a representation starting from the functional integrals for zero winding as obtained as obtained by Schroer and Nielsen [8]. In what follows we shall use the euclidean Dirac matrices

$$\gamma_1 = \begin{pmatrix} & -i \\ i & \end{pmatrix}, \qquad \gamma_2 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$$

and $\bar{\psi}$ will stand for $\psi^{\dagger}$.

It will be convenient to introduce the chiral densities

$$J_{\mp}(x) = N\left[\bar{\psi}\left(\frac{1 \pm \gamma^5}{2}\right)\psi\right](x). \tag{2.2}$$

As is well known from the operator solution [5], the chiral densities J_+ and J_- contain a spurion part which act as raising and lowering operators for the vacuum states $|\, n\rangle$, and only the products of fermion fields carrying chirality $2n$ will give rise to a non-vanishing matrix element in eq. (2.1). Introducing

$$\mathscr{G}(\xi; x, y) = \langle 0 \mid J_-(\xi_1) \cdots J_-(\xi_n)\, \psi_1(x_1) \cdots \psi_1(x_n)\, \bar{\psi}_1(y_1) \cdots \bar{\psi}_1(y_n) \mid 0\rangle \tag{2.3}$$

since we know that

$$\mathscr{G}(\xi; x, y) \xrightarrow[\xi \to \infty]{\xi_i = \xi + \eta_i} \langle 0 \mid \prod_{i=1}^{n} J_-(\xi + \eta_i) \mid n\rangle \times \langle n \mid \psi_1(x_1) \cdots \psi_1(x_n)\, \bar{\psi}_1(y_1) \cdots \bar{\psi}_1(y_n) \mid 0\rangle \tag{2.4}$$

we may extract in this limit the desired functional representation.

Our starting point is the euclidean generating functional for a $|\, 0\rangle \to |\, 0\rangle$ transition in the presence of external (anticommuting c-number) sources η and $\bar{\eta}$. In the Lorentz gauge we have, after integrating over the fermion fields

$$Z[\eta, \bar{\eta}] = N \int \mathscr{D}[A_T]\, e^{-S_0[A]}\, \mathrm{Det}[i(\not{\partial} - ie\not{A})]\, e^{-\int d^2x d^2y \bar{\eta}(x) G(x,y;A) \eta(y)} \tag{2.5}$$

where the integration is done over transversal field configurations, $\mathscr{D}[A_T]$ being short hand for $\mathscr{D}[A]\,\delta(\partial_\mu A_\mu)$. N and $S_0[A]$ denote the usual normalization constant and free electromagnetic action respectively, and $G(x, y; A)$ is the Greens function for the Dirac operator in an external field:

$$i(\not\partial - ie\not A)G(x, y; A) = \delta^2(x - y)$$

For the Schwinger model, $G(x, y; A)$ may be calculated explicitly,

$$G(x, y; A) = e^{ie(\phi(x)-\phi(y))}\, G_0(x - y) \tag{2.6a}$$

where $G_0(z)$ is the free Dirac Greens function, $\phi(x)$ is given by [8]

$$\phi(x) = -\int d^2z\, D(x - z)(\partial_\mu A_\mu(z) - i\gamma_5 \epsilon_{\mu\nu}\, \partial_\mu A_\nu(z)) \tag{2.6b}$$

and

$$D(x) = -\frac{1}{4\pi} \ln \mu^2 x^2$$

is the regulated zero-mass propagator.

The (gauge invariant) functional determinant is also explicitely calculable in this model:

$$\operatorname{Det} i(\not\partial - ie\not A) = \exp\left\{-\frac{e^2}{2\pi}\int d^2x\, A_\mu\left(\delta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\Box}\right)A_\nu\right\}. \tag{2.7}$$

From the generating functional (2.5) one obtains all non-tunneling fermion Greens functions by functional differentiation with respect to the sources. In particular, for the Green function (2.3) one has

$$\mathscr{G}(\xi; x, y) = N\int \mathscr{D}[A_T^{[0]}]\, e^{-\Gamma - S_0}\mathscr{G}(\xi; x, y; A) \tag{2.8a}$$

where the integration is done over zero-winding (transversal) fields,

$$\Gamma = \frac{e^2}{2\pi}\int d^2x\, A_\mu(z)\, A_\mu(z) \tag{2.8b}$$

is the logarithm of the determinant (2.7) (in the Schwinger gauge),

$$S_0 = -\frac{1}{2}\int d^2z\, A_\nu(z)\,\Box A_\nu(z) \tag{2.8c}$$

is the free electromagnetic action, and $\mathscr{G}(\xi; x, y; A)$ is the external field Greens function corresponding to (2.3); it is given as a product of the two-point Greens functions above and has the form

$$\mathscr{G}(\xi, x, y; A) = e^{-\Phi(\xi, x, y)}\, G_0(\xi; x, y) \tag{2.8d}$$

where

$$\Phi(\xi; x, y) = -e \int d^2z \sum_{i=1}^{n} (2D(\xi_i - z) - D(x_i - z) - D(y_i - z))\, \epsilon_{\lambda\mu}\, \partial_\lambda A_\mu(z). \tag{2.8e}$$

A convenient representation for $G_0(\xi; x, y)$ is given by

$$G_0(\xi; x, y) = \left(\frac{1}{2\pi}\right)^{2n} \prod_{j<k} \frac{(\xi_j - \xi_k)(x_j - x_k)_+(y_j - y_k)_-}{\prod_{j,k} (\xi_j - x_k)_+ \prod_{j,k} (\xi_j - y_k)_-} \tag{2.9}$$

where, typically

$$x_\pm = \pm x_1 - ix_2$$

As was shown in Ref. [9], the effective action in (2.8) is minimized by the (transversal) field configuration

$$A_\mu^{\mathrm{cl}}(z) = \frac{\pi}{e}\, \epsilon_{\lambda\mu} \frac{\partial}{\partial z_\lambda} \sum_{i=1}^{n} (2\mathscr{D}(\xi_i - z) - \mathscr{D}(x_i - z) - \mathscr{D}(y_i - z)) \tag{2.10}$$

where

$$\mathscr{D}(z) = D(z) - \Delta(z)$$

and $\Delta(z)$ is the euclidean propagator with a mass e^2/π. The configuration $(2\pi/e)$ $\epsilon_{\lambda\mu}\, \partial_\lambda\, \mathscr{D}(\xi - z)$ is an extended regular vortex with topological number -1 (induced instanton) peaked around $z = \xi$, where ξ is the variable that will be taken to infinity.

Expression (2.10) suggests making the following splitting of A_μ in eq. (2.5),

$$A_\mu(z) = \frac{2\pi}{e}\, \epsilon_{\lambda\mu} \frac{\partial}{\partial z_\lambda} \sum_{i=1}^{n} \mathscr{D}(\xi_i - z) + A_\mu^{[n]}(z) \tag{2.11}$$

where the new integration variable $A_\mu^{[n]}$ carries winding number n and has vanishing overlap with the extended vortex in the limit of large ξ. Introducing (2.11) into eq. (2.8) and taking the limit $\xi \to \infty$, we find

$$S_0 \to \frac{2\pi^2}{e^2} \int d^2z \sum_{j,k} \mathscr{D}(\xi_j - z)\, \Box^2 \mathscr{D}(\xi_k - z) + S_0' \tag{2.12}$$

where

$$S_0' = -\frac{1}{2} \int d^2z\, A_\mu^{[n]}(z)\, \Box A_\mu^{[n]}(z). \tag{2.13}$$

In estimating $\Phi(\xi; x, y)$ we find,

$$\begin{aligned} \Phi(\xi; x, y) \to\ & -n \ln \mu^2\xi^2 \cdot e \int \frac{d^2z}{2\pi}\, \epsilon_{\lambda\mu} \partial_\lambda A_\mu^{[n]} \\ & + 2\pi \sum_{k,i} (\mathscr{D}(\xi_k - x_i) + \mathscr{D}(\xi_k - y_i)) - 4\pi \sum_{k,i} \mathscr{D}(\xi_k - \xi_i) \\ & - e \int d^2z \sum_{i=1}^{n} (D(x_i - z) + D(y_i - z))\, \epsilon_{\lambda\mu} \partial_\lambda A_\mu^{[n]}(z) \end{aligned}$$

Since the integral multiplying the logarithm gives the topological number of $A_\mu^{[n]}$, we have in the limit,

$$\Phi(\xi; x, y) \to -2n^2 \ln \mu^2\xi^2 - 4\pi \sum_{k,i} \mathscr{D}(\xi_k - \xi_i) + \Phi'(x, y) \tag{2.14}$$

where $\Phi'(x, y)$ is Φ, reduced to the variables x, y:

$$\Phi'(x, y) = -e \int d^2z \sum_{i=1}^{n} (D(x_i - z) + D(y_i - z))\, \epsilon_{\lambda\mu}\partial_\lambda A_\mu^{[n]}(z) \tag{2.15}$$

In order to control the large ξ-limit of Γ it is convenient to split $A_\mu^{[n]}$ into any given vortex with winding number n and a remainder a_μ carrying zero winding:

$$A_\mu^{[n]}(z) = \frac{2n\pi}{e} \epsilon_{\lambda\mu}\partial_\lambda V(z) + a_\mu(z)$$

with

$$V(z) \xrightarrow[z\to\infty]{} \frac{1}{4\pi} \ln z^2$$

We find,

$$\begin{aligned}\Gamma \to n \ln \mu^2\xi^2 \cdot e \int \frac{d^2z}{2\pi} \epsilon_{\lambda\mu}\partial_\lambda A_\mu^{[n]} \\ - 2\pi \int d^2z \sum_{j,k} \mathscr{D}(\xi_j - z) \square \mathscr{D}(\xi_k - z) + \Gamma'\end{aligned} \tag{2.16}$$

where

$$\Gamma' = \frac{e^2}{2\pi} \int d^2z (A_\mu^{[n]}(z)\, A_\mu^{[n]}(z) - \left(\frac{2n\pi}{e}\right)^2 \partial_\nu(V(z)\, \partial_\nu V(z))) \tag{2.17}$$

Observe that the second term in the integrand of Γ' only depends on the asymptotic behaviours of V and ensures that the integral is finite. As will become clear in the following section, the subtraction term plays the role of omitting the zero eigenvalues in the fermion-determinant in the presence of winding potentials.

Adding the contributions for S_0, Φ and Γ, the resulting expression can be cast into the form,

$$S_0 + \Phi + \Gamma = -n^2 \ln \mu^2\xi^2 - 2n\pi\mathscr{D}(0) - 4\pi \sum_{j<k} \mathscr{D}(\xi_j - \xi_k) + S_0' + \Phi' + \Gamma' \tag{2.18}$$

Using the definition of $\mathscr{D}(z)$ and expression (2.9), we obtain upon substitution into (2.8),

$$\begin{aligned}\mathscr{G}(\xi; x, y) \xrightarrow[\xi\to\infty]{} N\left(\frac{i\mu}{2\pi}\right)^{2n} e^{2n\pi\mathscr{D}(0)} e^{-4\pi\Sigma_{j<k}\Delta(\xi_j - \xi_k)} \\ \times \int \mathscr{D}[A_T^{[n]}]\, e^{-\Gamma' - S_0' - \Phi'} \prod_{j<k} \mu(x_j - x_k)_+ \prod_{j<k} \mu(y_j - y_k)_-\end{aligned} \tag{2.19}$$

Recognizing that the ξ-dependent factor in (2.19) just corresponds to the ξ dependent matrix element in formula (2.4) as calculated using the explicit operator solution [5], we finally arrive at

$$\langle n \mid \psi_1(x_1) \cdots \psi_1(x_n)\, \bar{\psi}_1(y_1) \cdots \bar{\psi}_1(y_n) \mid 0\rangle$$
$$= N\left(\frac{\mu}{2\pi}\right)^n \int \mathscr{D}[A_T^{[n]}]\, e^{-\Gamma'-S_0'} e^{-\Phi'(x,y)} \prod_{j<k} \mu(x_j - x_k)_+ \prod_{j<k} \mu(y_j - y_k)_- \tag{2.20}$$

The x, y dependence of the integrand in (2.20) is precisely the one expected from the general 't Hooft mechanism [2] for the cancellation of the zeroes of the determinant, resulting in products of the zero-energy fermion eigenfunctions. This point will be clarified in the following section.

The kinematic factor in eq. (2.20) is readily identified with the product over eigenstates of angular momentum, and corresponds to the fact, that for any given winding n we have n independent zero-energy bound states (this follows from the Atiyah-Singer theorem [11]) with angular momentum up to $n - 1$.

The amplitude (2.20) contains the minimal number of fields needed to give a nonvanishing result in the $|0\rangle$ to $|n\rangle$ transition because of the chiral selection rule. By following the above procedure, one readily finds the Feynman path representation in the general case:

$$\langle n \mid \psi_1(x) \cdots \psi_1(x_{n+l})\, \psi_2(x_1') \cdots \psi_2(x_k')\, \bar{\psi}_1(y_1) \cdots \bar{\psi}_1(y_{n+k})\, \bar{\psi}_2(y_1') \cdots \bar{\psi}_2(y_l') \mid 0\rangle$$
$$= N\left(\frac{\mu}{2\pi}\right)^{n+l+k} \int \mathscr{D}[A_T^{[n]}]\, e^{-\Gamma'-S_0'-\Phi'(x,y;x',y')}\,.$$
$$\cdot \frac{\prod_{i<j} \mu(x_i - x_j)_+ \prod_{i<j} \mu(y_i' - y_j')_+ \prod_{i<j} \mu(y_i - y_j)_- \prod_{i<j} \mu(x_i' - x_j')_-}{\prod_{i,j} \mu(y_i' - x_j)_+ \prod_{i,j} \mu(x_i' - y_j)_-} \tag{2.21}$$

where S_0' and Γ' are the same as in eqs. (2.13) and (2.17), and

$$\Phi'(x, y; x', y') = \Phi'(x, y) - \Phi'(x', y')$$

with $\Phi'(x, y)$ given by eq. (2.15).

The case of negative winding numbers is obtained from eq. (2.21) by the simple replacements $1 \leftrightarrow 2$ and $z_+ \leftrightarrow z_-$. The structure of (2.21) becomes particularly translucent for $n = \pm 1$, where it can be cast into the form,

$$\langle \pm 1 \mid \psi_{\alpha_1}(x_1) \cdots \psi_{\alpha_l}(x_l)\, \bar{\psi}_{\beta_1}(y_1) \cdots \bar{\psi}_{\beta_l}(y_l) \mid 0\rangle$$
$$= N\left(\frac{\mu}{2\pi}\right) \int \mathscr{D}[A_T^{[\pm 1]}]\, e^{-\Gamma'-S_0'-\Phi'(x,y)}$$
$$\cdot \sum_P (-1)^P \left(\frac{1 \mp \gamma^5}{2}\right)_{\alpha_k \beta_{k'}} \prod_{\substack{i \neq k \\ j_i \neq k'}} G_0(x_i - y_{j_i})_{\alpha_i \beta_{j_i}} \tag{2.22}$$

where $G_0(x)$ is the free, massless fermion Green's function, and Φ' takes the form,

$$\Phi'(x, y) = e \sum_i \int d^2z [\gamma^5_{\alpha_i\alpha_i} D(x_i - z) + \gamma^5_{\beta_i\beta_i} D(y_i - z)] \, \epsilon_{\lambda\mu} \partial_\lambda A^{[\pm 1]}_\mu(z)$$

Thus, the net effect in going from winding zero to winding ± 1 consists in replacing one G_0 by the projector $\frac{1}{2}(1 \mp \gamma^5)$.

Finally we observe that, as expected from the work of Nielsen and Schroer [8], the effective action $\Gamma' + S_0' + \Phi'$ is minimized by an "induced instanton" carrying winding number n. Furthermore, since this effective action is quadratic in the fields, the stationary phase contribution gives the exact result.

3. Comparison with Perturbation Theory

We find it instructive to rederive our previous results from a perturbative point of view which will throw some light on the connection between the various factors appearing in our functional representation for non-trivial winding, and the structure provided by the general 't Hooft mechanism [2]. In order to keep the discussion as simple as possible we shall only consider the two-point function for winding number one, in lowest order perturbation theory.

In four-dimensional non-abelian gauge theories such a perturbative treatment would consist in expanding the classical gauge field configuration around the instanton [2]. In the Schwinger model, where there exist no finite action, non-trivial solutions to the pure gauge part, one will have to expand around any given configuration V_μ carrying winding number one

$$A^{[1]}_\mu = V^{[1]}_\mu + a_\mu$$

with

$$V^{[1]}_\mu = \frac{2\pi}{e} \epsilon_{\lambda\mu} \partial_\lambda V(z)$$

We then have,

$$\langle 1 \mid \psi_\alpha(x) \bar{\psi}_\beta(y) \mid 0 \rangle = N' \int \mathscr{D}[a_T] \, \mathscr{D}[\psi] \, \mathscr{D}[\bar{\psi}] \, \psi_\alpha(x) \bar{\psi}_\beta(y) \, e^{-S_0[A^{[1]}]}$$

$$\cdot \exp \left\{ - \int d^2z \, \bar{\psi}(z) \, \not{D} \psi(z) \right\} \left(1 + e \int d^2z \, \bar{\psi} \not{a} \psi + O(a^2) \right)$$

where

$$\not{D} = i\not{\partial} - e\not{V}^{[1]}$$

We will expand the fermion fields in terms of a suitable basis. Since the spectrum of the Dirac operator $\not{D}$ with respect to the usual measure is continuous, we find it useful to introduce as a basis the eigenfunction ϕ_k of $\not{D}$, orthonormal with respect to a suitable measure $\rho_R(z)$, which renders the spectrum discrete and has the property $\rho_R(z) \to 1$ as $R \to \infty$:

$$\not{D}\phi_k = \lambda_k \rho_R \phi_k$$

$$\int d^2z\, \rho_R(z)\, \bar{\phi}_k(z)\, \phi_{k'}(z) = \delta_{kk'}$$

A convenient choice could be $\rho_R(z) = (1 + z^2/R^2)^{-1}$, corresponding to the compactification of euclidean space into a sphere of radius R [11, 12].

In terms of the eigenfunctions ϕ_k we find,

$$\begin{aligned}\langle 1 \mid \psi_\alpha(x)\, \bar{\psi}_\beta(y) \mid 0\rangle = N \frac{\prod_{k\neq 0} \lambda_k}{\pi \lambda_k^{(0)}} \int \mathscr{D}[a]\, e^{-S_0[V^{[1]}+a]} \Big\{ &\phi_0(x)_\alpha\, \bar{\phi}_0(y)_\beta \\ &+ e\Big[\phi_0(x)_\alpha\, \bar{\phi}_0(y)_\beta \int d^2z\, \mathrm{tr}[G'(z, z)\, \not{a}(z)] \\ &+ \int d^2z (G'(x, z)\, \not{a}(z)\, \phi_0(z))_\alpha\, \bar{\phi}_0(y)_\beta \\ &+ \phi_0(x)_\alpha \int d^2z (\phi_0(z)\, \not{a}(z)\, G'(z, y))_\beta \\ &- \Big(\int d^2z\, \bar{\phi}_0(z)\, \not{a}(z)\, \phi_0(z)\Big)\, G'(x, y)_{\alpha\beta}\Big]\Big\} + O(a^2) \end{aligned} \quad (3.1)$$

where $\lambda_k^{(0)}$ are the eigenvalues of the free Dirac equation; $\phi_0(x)_\alpha$ is the zero energy eigenstate of $\not{D}$ ($\lambda_0 = 0$) and, up to normalization, is independent of ρ_R:

$$\phi_0(x) = \mathscr{N}_R^{1/2} \begin{pmatrix} \exp\Big\{e \int d^2z\, D(x - z)\, \epsilon_{\lambda\mu} \partial_\lambda V_\mu^{[1]}(z)\Big\} \\ 0 \end{pmatrix} \quad (3.2)$$

and

$$G'_{\alpha\beta}(x, y) = \sum_{k\neq 0} \frac{\phi_k(x)_\alpha\, \bar{\phi}_k(y)_\beta}{\lambda_k}. \quad (3.3)$$

One recognizes at once in the combination $\prod_{k\neq 0} \lambda_k \phi_0(x)_\alpha \bar{\phi}_0(y)_\beta$ the 't Hooft mechanism for massless fermions in the presence of the winding configuration $V_\mu^{[1]}$. The non-vanishing of the amplitude (3.1) is guaranteed by the fact that $S_0[V^{[1]}]$ is finite.

We now show that the first-order terms in eq. (3.1) correspond to the lowest order corrections to the fermion determinant and zero-energy wave functions, in accordance with the result of the previous section. The function $G'(x, y)$ satisfies

$$\not{D}_x G'(x, y) = \delta(x - y) - \rho_R(x)\, \phi_0(x)\, \bar{\phi}_0(y)$$

The normalization factor $\mathscr{N}_R$ in (3.2) goes to zero when $R \to \infty$, corresponding to the absence of bound states of the Dirac equation with respect to the unit measure. Hence $G'(x, y)$ becomes identical in this limit with the usual Green's function,

$$G(x, y) = \exp\Big\{-e\gamma^5 \int d^2z (D(x - z) - D(y - z))\, \epsilon_{\lambda\mu} \partial_\lambda V_\mu^{[1]}(z)\Big\} \cdot G_0(x - y). \quad (3.4)$$

with

$$\chi_0{}^E = -\frac{e^2(s - m^2 - m'^2)}{2p(s)^{1/2}} \int_{-\infty}^{\infty} V(b, z)\, dz \tag{34a}$$

$$\chi_2{}^E = \frac{e^4(m + m')}{2p(s)^{1/2}} \int_{-\infty}^{\infty} V^2(b, z)\, dz, \chi_1{}^E = -\frac{(s - m^2 - m'^2)^2}{4p^2 s}\left(1 + b\frac{\partial}{\partial b}\right)\chi_2{}^E,$$

$$[\gamma \times \gamma] = \bar{u}(p_2)\,\gamma_\mu u(p_1)\,\bar{u}(p_2')\,\gamma^\mu u(p_1'), \tag{34b}$$

and $V(b, z)$ as in Eq. (30b).

In the infrared limit we have now, in contrast with the scalar-scalar case, a non-vanishing first order correction

$$\operatorname*{Lt}_{\lambda\to 0} M^E = -e^2[\gamma \times \gamma] \exp\{-i\eta_E(2E_c - \ln(-t/\lambda^2))\}$$
$$\times \left[\frac{1}{t}\frac{\Gamma(1 - i\eta_E)}{\Gamma(1 + i\eta_E)} + \frac{e^2(m + m')}{8(s - m^2 - m'^2)}\frac{1}{(-t)^{1/2}}\frac{\Gamma(\frac{1}{2} - i\eta_E)}{\Gamma(\frac{1}{2} + i\eta_E)}\right], \tag{35a}$$

where

$$\eta_E = -\frac{e^2(s - m^2 - m'^2)}{8\pi p s^{1/2}}, \tag{35b}$$

and E_c, as before, is the Euler's constant. The second term in the square bracket in Eq. (35a) is of $O(m(-t)^{1/2}/s)$ relative to the first and hence, according to Eq. (26), a bonafide first order correction. It may be observed that for fermion-antifermion scattering η_E changes sign and the poles of the Γ-functions in (35a) appear on the physical sheet. Whereas the poles of $\Gamma(1 - i\eta_E)$ in the leading term may be related to Coulomb bound states [18] those of $\Gamma(\frac{1}{2} - i\eta_E)$ in the second term do not have such analogues. It should, however, be pointed out that in the neighbourhood of these poles the eikonal phase provide extra powers (positive integral or half-integral) of t and our amplitude ceases to be singular at $t = 0$. In the neighbourhood of these poles the terms ignored in eikonal approximation is just as important as the leading term in Eq. (35a) and the bonafide of the poles is, therefore, an open question. The correction term in (35a) has also been obtained recently by Bazhanov *et al.* [19] by dispersion theoretic method. The static limit ($m' \to \infty$, say) of (35) gives the Coulomb amplitude [20] for a relativistic Dirac particle.

IV. Fixed Angle Behaviour

4.1. *Leading Infrared Behaviour*

All the infrared scalings appropriate for the fixed angle domain (s, $|t|$, $m^2 \gg \lambda^2$) have been obtained in sect. II. There we learnt that the leading infrared divergence of a diagram with exchanged mesons is built up of contributions from the leading meson out, however, that one can still extract the tunneling amplitudes from $\mathscr{G}(y_1; x, y)$ in the limit that $y_1 \to \infty$. To show this, it is convenient to write the ψ-field in bosonized form (up to a Klein transformation)

$$\psi_\alpha = \left(\frac{\mu}{2\pi}\right)^{1/2} e^{i(\pi)^{1/2}\gamma^5_{\alpha\alpha}(\tilde{\Sigma}+\tilde{\eta})} :e^{i(\pi)^{1/2}(\gamma^5_{\alpha\alpha}\tilde{\phi}+\phi)}: \hat{\sigma}_\alpha \tag{4.1}$$

where η and ϕ are infrared regularized massless fields, η is quantized with the opposite sign in the commutation relations, $\tilde{\Sigma}$ is a free pseudoscalar field of mass e^2/π and $\hat{\sigma}_\alpha$ is a spurion whose sole purpose is to carry the fermionic selection rules.[2]

The vacuum states of the theory are known to be

$$| n_1, n_2\rangle = \hat{\sigma}_1^{n_1}\hat{\sigma}_2^{n_2} | 0\rangle.$$

Using the operator solution (4.1), we explicitely verify (with $y_1 = \xi$) that

$$\lim_{\xi\to\infty} \frac{i\mu\xi_+}{(\mu\xi^2)^{1/4}} \mathscr{G}(\xi; x, y) = \langle 0 | \bar{\psi}_2 | 1, 0\rangle\langle 1, 0 | \bar{\psi}(y) \cdots \psi(x_n) | 0\rangle \tag{4.2}$$

Expressing $\mathscr{G}(\xi; x, y)$ as a Feynman-path integral, we arrive at

$$\begin{aligned} &\langle 1, 0 | \bar{\psi}_{\beta_2}(y_2) \cdots \bar{\psi}_{\beta_n}(y_n)\, \psi_{\alpha_1}(x_1) \cdots \psi_{\alpha_n}(x_n) | 0\rangle \\ &= N\left(\frac{\mu}{2\pi}\right)^{1/2} \int \mathscr{D}[A_T^{[1/2]}]\, e^{-\Gamma'-S'-\Phi'(x,y)} \\ &\cdot \sum_P (-1)^P \delta_{\alpha_k,1} \prod_{i\neq k} G_0(x_i - y_{j_i}) \end{aligned} \tag{4.3}$$

where

$$\Phi'(x, y) = e\int d^2z \left[\sum_{i=1}^n \gamma^5_{\alpha_i\alpha_i} D(x_i - z) + \sum_{j=2}^n \gamma^5_{\beta_j\beta_j} D(y_i - z)\right] \epsilon_{\lambda\mu}\partial_\lambda A_\mu^{[1/2]}(z).$$

The effective action in (4.3) is minimized by a configuration carrying a half-unit of topological charge. For instance, for $n = 1$, we have

$$A_\mu^{cl}(z) = -\frac{\pi}{e}\epsilon_{\lambda\mu}\frac{\partial}{\partial z_\lambda}\mathscr{D}(x - z).$$

The fact that pairs of such configurations saturate the two-point function, was

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tation, which is geared to canonical fields, for amplitudes involving $\hat{\psi}$. Indeed, in the transverse gauge, in which we have been working, $A_\mu{}^L = 0$, we are certainly not going to reproduce the well known amplitudes involving $\hat{\psi}$ via a path integral. By contrast in the operator solution, although A_μ in the Schwinger gauge is transversal, it nevertheless has a massless part which can be identified with $A_\mu{}^L$. On the other hand, to work with a Mandelstam field, where the full A_μ appears in the line integrals, is inappropriate, since this field is ill-defined in the Schwinger model.

[2] In previous works [10, 13] the exponentials of ϕ themselves were the carriers of the selection rules, which here we choose to make explicit by means of $\hat{\sigma}$.

already noted by Nielsen and Schroer [9]. Here we have been able to isolate the individual contributions which play with respect to the half-integer gauge classes introduced in [10] the same role, as the instanton does with respect to the integer gauge classes.

5. Conclusions

In this work we have used clustering and generalized clustering arguments to derive, starting from the functional representations of Nielsen and Schroer for winding number zero, Feynman-path integrals for tunneling amplitudes involving integer and half-integer winding configurations.

We also exibited the relation between the path-integrals with integer winding configurations thus obtained and the general form expected on the basis of 't Hooft's mechanism for fermions in the presence of winding fields.

After the completion of this work we became aware of B. Schroer's Schladming lectures where path integral representations over non-trivial winding are also written down.

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